SHOCK WAVE FROM A SLIGHTLY CURVED PISTON

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The method of solution of the problem already discussed in [1] is recommended. This method does not require the construction of "conical solutions". It allows one to solve other problems which reduce to a hyperbolic system of equations with boundary conditions on the moving boundaries.

1. Formulation of the problem. In an undisturbed gaseous medium we assume the YZ plane to coincide with the surface of the piston; at instant $t = 0$ the piston has started to move along the X axis at constant velocity U; a shock wave travels through the gas at velocity D. In the initial state the gas density is ρ_0, the velocity of sound is c_0, whilst behind the wave-front they are ρ, c, respectively. We assume the gas to be an ideal one with isentropic index γ. The velocity of the wave with respect to the piston is denoted by V, so that $D = U + V$. Introduce parameter $\delta = 1/M_0^2$ where $M_0 = D/c_0$. We then have the known expressions

$$\sigma = \frac{\rho}{\rho_0} = \frac{h}{1+(h-1)\delta}, \qquad V = \frac{D}{\sigma}, \qquad \beta^2 = \frac{V^2}{c^2} = \frac{1+(h-1)\delta}{(h+1)-\delta}, \qquad h = \frac{\gamma+1}{\gamma-1}$$

Having obtained the undisturbed solution, we deal with the propagation of the shock wave from a slightly curved piston as a linear approximation. Without losing generality we may consider the piston surface to be bent in one direction (only) and to be in the form $\epsilon(Y)$. We employ a system of coordinates in which the piston is at rest. Within the region $0 < X < Vt$ we have the following linearised equations for the pressure disturbance p' and the velocity components v_x' and v_y'

$$\frac{\partial p'}{\partial t} + \rho c^2\left(\frac{\partial v_x'}{\partial X} + \frac{\partial v_y'}{\partial Y}\right) = 0, \qquad \frac{\partial v_x'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial X} = 0, \qquad \frac{\partial v_y'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial Y} = 0 \qquad (1.1)$$

Changes in density ρ' are eliminated by using the adiabatic condition

$$\frac{\partial p'}{\partial t} = c^2 \frac{\partial \rho'}{\partial t}$$

The boundary condition at the wall will equate the normal velocity component to zero, $v_x' = 0$, (for the linear approximation with $X = 0$). In accordance with the second of the equations (1.1) we also have $\partial p'/\partial X = 0$. The condition at the front of the shock wave is derived from [2]. Note that, for an ideal gas, the following is valid:

$$-j^2 \left[\frac{\partial}{\partial p}\left(\frac{1}{\rho}\right)\right]_H = \delta < 1$$

Using the conventional notation we have

$$v_y' = -U \frac{\partial \xi}{\partial Y}, \quad v_x' = \frac{1+\delta}{2\rho_0 D} p', \quad \frac{\partial \xi}{\partial_t} = \frac{1-\delta}{2\rho_0 U} p' \quad \text{when } X = Vt \qquad (1.2)$$

The displacement of the wave-front from the plane $X = Vt$ is denoted by $\xi(Y, t)$. To eliminate $\xi(Y, t)$ from the boundary conditions we differentiate the first of the equations (1.2) with respect to time:

$$\frac{dv_y'}{dt} = \frac{\partial v_y'}{\partial t} + V \frac{\partial v_y'}{\partial_X} = -U \frac{\partial^2 \xi}{\partial_Y \partial t}$$

Using (1.1) and (1.2) we obtain

$$V \frac{\partial v_y'}{\partial X} = \left(\frac{1}{\rho} - \frac{1-\delta}{2\rho_0}\right) \frac{\partial p'}{\partial Y} \qquad \text{when } X = Vt$$

The initial conditions come from the fact that the wave-front when $t = 0$ coincides with the surface of the piston where $v_x' = 0$ always. In accordance with (1.2) $p' = 0$ when $t = 0$. The velocity component v_y' at the initial instant is not zero, but is given by $v_y'(0) = -U d\epsilon / dY$.

Let $\epsilon(Y) = \Delta \exp(ikY)$, where Δ and k are both constant, and $k\Delta \ll 1$ (small disturbance or displacement). The dependence of all quantities on the coordinate Y is then expressed by the multiplier $\exp(ikY)$. Introduce the following notations:

$$p' / \rho c = w, \qquad v_x' = u, \qquad v_y' = -iv$$

We also make the following transformation:

$$kX = x, \qquad kct = y$$

The problem then reduces to solving the system

$$\frac{\partial w}{\partial y} + \frac{\partial u}{\partial x} + v = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} - w = 0 \qquad (1.3)$$

with boundary conditions

$$u=0,\ \frac{\partial w}{\partial x}=0 \text{ when } x=0; \qquad u=Aw,\ \frac{\partial v}{\partial x}=Bw \text{ when } x=\beta y\ (\beta<1) \tag{1.4}$$

where

$$A=\frac{1+\delta}{2\beta}, \qquad B=\frac{1}{\beta}\left[\frac{1-\delta}{2}\frac{\rho}{\rho_0}-1\right]$$

and initial conditions

$$u=w=0, \quad v=v_0 \quad \text{when } x=y=0 \quad (v_0=Uk\Delta) \tag{1.5}$$

Note that that function $w(x, y)$ satisfies the equation

$$\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}+w=0 \tag{1.6}$$

2. Solution of the boundary-value problem. Introduce the new variables r and θ using the formulas

$$y=r\cosh\theta, \qquad x=r\sinh\theta, \qquad r=\sqrt{y^2-x^2}, \qquad \tanh\theta=\frac{x}{y} \tag{2.1}$$

We multiply the first of Equations (1.3) by $\cosh\theta$ and add it to the second multiplied by $\sinh\theta$; we then interchange the positions of $\cosh\theta$ and $\sinh\theta$. This results, finally, in the system

$$\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}+v\cosh\theta=0, \qquad \frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial\theta}+v\sinh\theta=0$$
$$\operatorname{ch}\theta\frac{\partial v}{\partial r}-\frac{\partial v}{\partial\theta}\frac{\sinh\theta}{r}-w=0 \tag{2.2}$$

Equation (1.6) is transformed into the form

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+w=0 \tag{2.3}$$

The line $x=0$ corresponds to $\theta=0$, line $x=\beta y$ corresponds to $\theta=\theta_0$ where $\tanh\theta_0=\beta$. In the third equation of the system (2.2) we will put $\theta=\theta_0$ and to it we will add the product of $\partial v/\partial x=Bw$ and $\tanh\theta_0$, which also holds along $\theta=\theta_0$. The origin of coordinates $x=y=0$ is given by $r=0$. As a result, the boundary conditions and the initial conditions take the following form:

$$u=0, \qquad \frac{\partial w}{\partial\theta}=0, \qquad \text{at } \theta=0 \tag{2.4}$$

$$u=Aw, \qquad \frac{\partial v}{\partial r}=(B\sinh\theta_0+\cosh\theta_0)\,w \qquad \text{at } \theta=\theta_0 \tag{2.5}$$

$$u=w=0, \quad v=v_0 \qquad \text{at } r=0 \tag{2.6}$$

Now let us make use of the Laplace transformation for the variable r according to the formula

$$f_1(p, \theta) = \int_0^\infty e^{-pr} f(r, \theta)\, dr$$

We then obtain

$$\frac{\partial}{\partial p}(pw_1) - \frac{\partial u_1}{\partial \theta} + \cosh\theta \frac{\partial v_1}{\partial p} = 0, \qquad \frac{\partial}{\partial p}(pu_1) - \frac{\partial w_1}{\partial \theta} + \sinh\theta \frac{\partial v_1}{\partial p} = 0$$

$$\cosh\theta \frac{\partial}{\partial p}(pv_1) + \sinh\theta \frac{\partial v_1}{\partial \theta} - \frac{\partial w_1}{\partial p} = 0 \qquad (2.7)$$

$$(p^2 + 1)\frac{\partial^2 w_1}{\partial p^2} + 3p\frac{\partial w_1}{\partial p} + w_1 - \frac{\partial^2 w_1}{\partial \theta^2} = 0 \qquad (2.8)$$

$$\begin{aligned} &u_1 = 0, \qquad \frac{\partial w_1}{\partial \theta} = 0 && \text{at } \theta = 0 \\ &u_1 = Aw_1, \quad pv_1 - v_0 = (B\sinh\theta_0 + \cosh\theta_0)\, w_1 && \text{at } \theta = \theta_0 \end{aligned} \qquad (2.9)$$

Besides it follows from the Laplace transformation theory that all transformed functions should fulfil the condition

$$f_1(p, \theta) \to 0 \qquad \text{at } \operatorname{Re} p \to +\infty \qquad (2.10)$$

Make the substitution

$$p = \sinh q, \qquad w_1(p, \theta) = w_2(q, \theta)/\cosh q$$

Then, instead of obtaining (2.8) for function $w_2(q, \theta)$, we get

$$\frac{\partial^2 w_2}{\partial q^2} - \frac{\partial^2 w_2}{\partial \theta^2} = 0$$

The general solution of this wave equation is of the form

$$w_2(q, \theta) = F(q + \theta) + \Phi(q - \theta)$$

where F and Φ are arbitrary functions. It is clear from (2.9) that $\partial w_2/\partial\theta = 0$ when $\theta = 0$, therefore $\Phi(q) = F(q)$ and

$$w_2(q, \theta) = F(q + \theta) + F(q - \theta) \qquad (2.11)$$

The second of Equations (2.7) can be written thus:

$$\frac{\partial}{\partial q}\{pu_1 - [F(q + \theta) - F(q - \theta)] + \sinh\theta\, v_1\} = 0$$

From this we get

$$pu_1 - [F(q+\theta) - F(q-\theta)] + \sinh\theta v_1 = \varphi(\theta) \tag{2.12}$$

where $\phi(\theta)$ is an arbitrary function. It is known [3] that

$$f(r=0) = f(0) = \lim pf_1(p, \theta) \quad \text{for } p \to \infty$$

It follows, therefore, that

$$w(0) = \lim_{p\to\infty} pw_1(p, \theta) = 0, \qquad \lim_{q\to\infty} \tanh q w_2(q, \theta) = 2F(\infty) = 0$$
$$u(0) = \lim_{p\to\infty} pu_1 = 0, \qquad \lim_{q\to\infty} pu_1 = 0 \tag{2.13}$$

In accordance with (2.10) we also have $\lim v_1 = 0$ when $p \to \infty$ and $\lim v_1 = 0$ when $q \to \infty$. In Equation (2.12) we turn our attention to $\mathrm{Re}\, q \to \infty$. Then, the L.H.S. vanishes for any value of θ, i.e. $\phi(\theta) \equiv 0$ and

$$pu_1 - [F(q+\theta) - F(q-\theta)] + \sinh\theta v_1 = 0$$

Here let us put $\theta = \theta_0$. Making use of (2.9) we find that $F(q)$ satisfies the finite difference equation

$$\sinh 2q [F(q+\theta_0) - F(q-\theta_0)] - (a\cosh 2q + b)[F(q+\theta_0) + F(q-\theta_0)] =$$
$$= 2v_0 \sinh\theta_0 \cosh q$$

where

$$a = A = \frac{1+\delta}{2\beta} > 1, \qquad b = 2\sinh\theta_0 (B\sinh\theta_0 + \cosh\theta_0) - A = \frac{1-\delta}{2\beta} \tag{2.14}$$

It is known [4] that the general solution of a non-homogeneous linear finite difference equation is the sum of the general solution of the homogeneous equation plus a particular solution of the equation including its R.H.S. The general solution of the homogeneous equation is proportionately a periodic function. In Equation (2.14) this period is $2\theta_0$. The uniqueness of the solution of (2.14) is insured by the circumstance that $F(q) \to 0$ when $\mathrm{Re}\, q \to \infty$ in accordance with (2.13). The homogeneous equation corresponding to (2.14), for sufficiently high values of $\mathrm{Re}\, q$, takes the form

$$F(q+\theta_0) = -\frac{a+1}{a-1} F(q-\theta_0)$$

i.e. with increase in $\mathrm{Re}\, q$ function, $F(q)$ grows indefinitely. Therefore, condition (2.13) can only be satisfied if we equate the arbitrary periodic multiplier to zero. The particular solution, which vanishes when $\mathrm{Re}\, q \to \infty$, takes the form of a series.

$$F(q)=\sum_{n=0}^{\infty} A_n e^{-(2n+1)q} \tag{2.15}$$

Put this expression into (2.14). Equating coefficients of similar powers, to evaluate the quantities $B_n = 2A_n \cosh(2n+1)\theta_0$ we arrive at the expression

$$B_0=-2v_0\frac{\sinh\theta_0}{a+\tanh\theta_0}, \qquad B_1=B_0\frac{(a-2b)+\tanh\theta_0}{a+\tanh 3\theta_0} \tag{2.16}$$

$$[a-\tanh(2n-1)\theta_0]B_{n-1}+2bB_n+[a+\tanh(2n+3)\theta_0]B_{n+1}=0 \quad (n=1,2,3,\ldots)$$

In accordance with (2.11) we have

$$w_2(q,\theta)=\sum_{n=0}^{\infty} B_n\frac{\cosh(2n+1)\theta}{\cosh(2n+1)\theta_0}e^{-(2n+1)q} \tag{2.17}$$

To demonstrate the convergence of the solution obtained, we examine those values of n in Equation (2.16) for which $2n\theta_0 \gg 1$. Then instead of (2.16) we get

$$(a-1)B_{n-1}+2bB_n+(a+1)B_{n+1}=0 \tag{2.18}$$

The solution of this difference equation with constant coefficients has the form $B_n = \mu^n$. The value of μ can be found from the quadratic

$$(a+1)\mu^2+2b\mu+(a-1)=0 \tag{2.19}$$

whose roots are negative and of absolute value less than unity. In accordance with Poincaré's theorem [4] when Re $q \geqslant 0$ the following series converges:

$$w_2(q,\theta_0)=\sum_{n=0}^{\infty} B_n e^{-(2n+1)q} \tag{2.20}$$

and, with it also, the series (2.17) for any values of $\theta \leqslant \theta_0$.

Now, returning to the variable p, we bear in mind that if $p = \sinh q$ then $\cosh q = \sqrt{p^2+1}$ and $e^{-q} = \sqrt{p^2+1} - p$; we have

$$w_1(p,\theta)=\sum_{n=0}^{\infty} B_n\frac{\cosh(2n+1)\theta}{\cosh(2n+1)\theta_0}\frac{(\sqrt{p^2+1}-p)^{2n+1}}{\sqrt{p^2+1}}$$

Using the known formula for representing Bessel functions [3]

$$J_n(r) \div \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$$

we obtain

$$w(r, \theta) = \sum_{n=0}^{\infty} B_n \frac{\cosh (2n+1)\theta}{\cosh (2n+1)\theta_0} J_{2n+1}(r) \qquad (2.21)$$

Now, to return to the original variables X, ct, we use the formulas

$$r = kct\sqrt{1-\tau^2}, \qquad \tau = \frac{X}{ct} \qquad (2.22)$$

$$\cosh (2n+1)\theta = \frac{1}{2}[(1+\tau)^{2n+1} + (1-\tau)^{2n+1}](1-\tau^2)^{-(n+1/2)}$$

The pressure at the front of the shock wave is given by the expansion

$$w(r, \theta_0) = \sum_{n=0}^{\infty} B_n J_{2n+1}(s), \qquad s = kct\sqrt{1-\beta^2} \qquad (2.23)$$

Let us insert (2.23) into the second of Equations (2.5) and integrate:

$$v(r, \theta_0) = v_0 + (B \sinh \theta_0 + \cosh \theta_0) \sum_{n=0}^{\infty} B_n \int_0^s J_{2n+1}(x)\,dx$$

It is known that for any value of n the integral on the R.H.S. when $s \to \infty$ equals unity. Note also the relationship which is obtained from (2.14) and (2.20) when $q = 0$:

$$w_2(0, \theta_0) = Q = \sum_{n=0}^{\infty} B_n = -\frac{2v_0 \sinh \theta_0}{a+b} = -\frac{v_0}{B \sinh \theta_0 + \cosh \theta_0} \qquad (2.24)$$

Bearing in mind that the quantities $v(r, \theta_0)$ and $\xi(s)$ are proportional to each other we arrive at the following expression for the shock-wave front:

$$\frac{v(r, \theta_0)}{v_0} = \frac{\xi(s)}{\Delta} = 1 - \frac{1}{Q}\sum_{n=0}^{\infty} B_n \int_0^s J_{2n+1}(x)\,dx = \frac{1}{Q}\sum_{n=0}^{\infty} B_n \int_s^{\infty} J_{2n+1}(x)\,dx \qquad (2.25)$$

It can be seen from this that $\xi(s) \to 0$ when $s \to \infty$.

The latter result can be expressed in a form without integrals

$$\frac{\xi(s)}{\Delta} = J_0(s) - \frac{1}{Q}\sum_{n=1}^{\infty} D_n J_{2n}(s) \qquad (2.26)$$

where

$$D_n = \frac{1}{a+b}\{[a + \tanh(2n+1)\theta_0] B_n - [a - \tanh(2n-1)\theta_0] B_{n-1}\} \qquad (2.27)$$

3. Certain limiting cases. To obtain asymptotic formulas for $r \gg 1$ we use the expression

$$J_{2n+1}(r) \sim (-1)^n \sqrt{\frac{2}{\pi r}} \left[\sin\left(r - \tfrac{1}{4}\pi\right) + \frac{4(2n+1)^2 - 1}{8r} \cos\left(r - \tfrac{1}{4}\pi\right)\right] \qquad (3.1)$$

and the relation

$$\sum_{n=0}^{\infty} (-1)^n B_n = 0 \qquad (3.2)$$

which is obtained from (2.14) putting $q = 1/2\, i\pi$ and $a \neq b$.

Using (3.1) and (2.21) we obtain

$$w(r, \theta) \sim \sqrt{\frac{2}{\pi r}} \sin\left(r - \frac{1}{4}\pi\right) \sum_{n=0}^{\infty} (-1)^n B_n \frac{\cosh(2n+1)\theta}{\cosh(2n+1)\theta_0} +$$

$$+ \frac{\cos(r - {}^1/_4\pi)}{4\sqrt{2\pi r^3}} \sum_{n=0}^{\infty} (-1)^n B_n \frac{\cosh(2n+1)\theta}{\cosh(2n+1)\theta_0} [4(2n+1)^2 - 1] \qquad (3.3)$$

At the shock-wave front, in view of (3.2), the first summation vanishes, so that there remains

$$w(r, \theta_0) \sim N \frac{\cosh(s - {}^1/_4\pi)}{\sqrt{2\pi s^3}} \qquad \left(N = 4 \sum_{n=0}^{\infty} (-1)^n n(n+1) B_n\right) \qquad (3.4)$$

With a strong shock-wave ($c_0 = 0$ or $U \to \infty$) $\delta = 0$ and $a = b$. One of the roots of Equation (2.19) becomes (-1), so that the series (3.4) diverges. We conclude from this that the asymptotic behaviour will differ substantially, namely, decay will be slower. In this case Equation (2.14) takes the form

$$\sinh q [F(q + \theta_0) - F(q - \theta_0)] - a \cosh q [F(q + \theta_0) + F(q - \theta_0)] = v_0 \sinh\theta_0 \qquad (3.5)$$

As before, we look for a solution in the form of (2.15), and instead of obtaining (2.16) for B_n we have the expression

$$B_0 = -\frac{2v_0 \sinh\theta_0}{a + \tanh\theta_0}, \qquad [a - \tanh(2n-1)\theta_0] B_{n-1} + [a + \tanh(2n+1)\theta_0] B_n = 0$$

$$(n = 1, 2, 3, \ldots) \qquad (3.6)$$

Convergence of the series (2.20) is obvious with such coefficients.

Because $B_n/B_{n-1} < 0$ for all values of n, in contradistinction to (3.2), we get

$$iw_2\left(\tfrac{1}{2}i\pi,\ \theta_0\right) = \sum_{n=0}^{\infty}(-1)^n B_n = \tfrac{1}{2}M \neq 0 \tag{3.7}$$

With the help of (3.3), with $\delta = 0$ we arrive at the following formula:

$$w(r,\ \theta_0) = M\,\frac{\sin\left(s - {}^1\!/_4\,\pi\right)}{\sqrt{2\pi s}} \tag{3.8}$$

Now we will obtain a formula which generalises (3.4) and (3.8) for the case of small but finite values of δ. Using integral representations of Bessel functions

$$J_{2n+1}(r) = \frac{2}{\pi}\,\mathrm{Im}\left[\int_0^{{}^1\!/_2 i\pi} e^{-(2n+1)q}\sinh(r\sinh q)\,dq\right]$$

instead of (2.23) we get

$$w(r,\ \theta_0) = \frac{2}{\pi}\,\mathrm{Im}\left[\int_0^{{}^1\!/_2 i\pi} w_2(q,\ \theta_0)\sinh(s\sinh q)\,dq\right] \tag{3.9}$$

Furthermore, from (2.14) it follows that

$$w_2(q,\ \theta_0) = \frac{2\cosh q}{\sinh 2q + a\cosh 2q + b}\left\{2\sinh q\,F(q+\theta_0) - v_0\sinh\theta_0\right\} \tag{3.10}$$

It is evident that for $q \to 1/2\,i\pi$ and $\delta \to 0$ the denominator of (3.10) vanishes and the main contribution to integral (3.9) for $s \gg 1$ is the point $q = 1/2\,i\pi$, whilst the expression in the curly brackets in the last formula can be replaced by its value for $q = 1/2\,i\pi$ and $\delta = 0$.

$$\begin{aligned} 2iF\left(\tfrac{1}{2}i\pi+\theta_0\right) - v_0\sinh\theta_0 &= i\left[F\left(\tfrac{1}{2}i\pi+\theta_0\right) - F\left(\tfrac{1}{2}i\pi-\theta_0\right)\right] - v_0\sinh\theta_0 + \\ &+ i\left[F\left(\tfrac{1}{2}i\pi+\theta_0\right) + F\left(\tfrac{1}{2}i\pi-\theta_0\right)\right] = \tfrac{1}{2}M \end{aligned} \tag{3.11}$$

The first two bracketed terms vanish because of the relation

$$i\left[F\left(\tfrac{1}{2}i\pi+\theta_0\right) - F\left(\tfrac{1}{2}i\pi-\theta_0\right)\right] = v_0\sinh\theta_0$$

which follows from (3.5) for $q = 1/2\,i\pi$. Note that in Formula (3.4) the coefficient N will be equal to

$$N = i\,\frac{\partial^2}{\partial q^2}\,w_2(q,\ \theta_0) \quad \text{when } q = \tfrac{1}{2}i\pi$$

After double differentiation we have from Equation (3.10)

$$N = -\frac{4M}{(h+1)\delta^2} \qquad \text{(for } \delta \ll 1) \tag{3.12}$$

Thus, the series (3.5) diverges as δ^{-2}. Using (3.9) and (3.10) we find that for $s \gg 1$ and $\delta \ll 1$

$$w(r, \theta_0) \approx \frac{2M}{\pi} \int_0^{1/2\pi} \frac{\sin(s \sin\varphi)\cos\varphi \sin 2\varphi}{(a\cos 2\varphi + b)^2 + (\sin 2\varphi)^2} d\varphi$$

Now, let us make the substitution $\sin\phi = x$. The main contribution in the integral comes from the neighbourhood of point $x = 1$, and, therefore, on changing the lower limit of integration to $-\infty$ where at all possible we put $x = 1$

$$w(r, \theta_0) \sim M \frac{4\sqrt{2}}{\pi} \int_{-\infty}^{1} \frac{\sin(sx)\sqrt{1-x}\, dx}{(a-b)^2 + 8(1-x)} \tag{3.13}$$

On putting $\alpha = 1/8(h+1)\, s\delta^2$, $1 - x = 1/8(h+1)\, z\delta^2$, we arrive at the required formula

$$w(r, \theta_0) \sim \frac{M\sqrt{h+1}\,\delta}{4\pi} \operatorname{Im}\{\psi(\alpha)\exp[i(s - \tfrac{1}{4}\pi)]\}$$

$$\psi(\alpha) = \exp(\tfrac{1}{4} i\pi) \int_0^{\infty} e^{-i\alpha z} \frac{\sqrt{z}\, dz}{z+1} \tag{3.14}$$

Note that function $\phi(\alpha)$ can be represented thus:

$$\psi(\alpha) = \sqrt{\frac{\pi}{\alpha}}\left[1 - \sqrt{\pi\alpha}\, e^{i\alpha}\left(e^{1/4 i\pi} - \frac{2i}{\sqrt{\pi}} \int_0^{\sqrt{\alpha}} e^{-i\eta^2} d\eta\right)\right] \tag{3.15}$$

If, in (3.13), the magnitude of s is fixed, and $\delta \to 0$, then $\alpha \to 0$ and Formula (3.13) transforms into (3.8). If δ is small, but fixed, whilst $s \to \infty$, then $\alpha/ \to \infty$ and we arrive at (3.4) taking into account (3.12).

The asymptotic behaviour of the function $\Delta^{-1}\xi(s)$ is established by substituting (3.4) and (3.8) in the second of Formulas (2.5):

$$\frac{\xi(s)}{\Delta} \sim \frac{N}{v_0} \frac{1}{2\beta \sinh\theta_0} \frac{\sin(s - 1/4\,\pi)}{\sqrt{2\pi s^3}} \qquad \text{when } \delta \neq 0$$

$$\frac{\xi(s)}{\Delta} \sim -\frac{M}{v_0} \frac{1}{2\sqrt{h(h+1)}} \frac{\cos(s - 1/4\,\pi)}{\sqrt{2\pi s}} \qquad \text{when } \delta = 0 \tag{3.16}$$

Note that with a strong shock wave ($\delta = 0$), in view of (3.6), Formula (2.27) reduces to the form $\alpha D_n = [\alpha + \tanh(2n+1)\theta_0]B_n$.

Formula (3.13) for $s \gg 1$ can be written thus

$$w(r, \theta_0) \sim -M \frac{4\sqrt{2}}{\pi} \frac{\partial}{\partial s} \left[\int_{-\infty}^{1} \frac{\cos(sx)\sqrt{1-x}\,dx}{(a-b)^2 + 8(1-x)} \right] \tag{3.17}$$

In a similar manner to (3.15) we find

$$w(r, \theta_0) \sim -\frac{M\sqrt{h+1}\delta}{4\pi} \frac{\partial}{\partial s} \operatorname{Re}\{\exp[i(s - \tfrac{1}{4}\pi)]\,\psi(\alpha)\}$$

Substituting in (2.5) we arrive at the formula

$$\frac{\xi(s)}{\Delta} \sim -\frac{M}{v_0} \frac{\delta}{8\pi\sqrt{h}} \operatorname{Re}\{\exp[i(s - \tfrac{1}{4}\pi)]\psi(\alpha)\} \tag{3.18}$$

If we make use of the Frennel tabulated integrals

$$S(z) = \int_0^z \sin t^2\,dt, \qquad C(z) = \int_0^z \cos t^2 dt$$

we will write (3.18) in the form

$$\frac{\xi(s)}{\Delta} \sim -\frac{M}{v_0} \frac{1}{2\sqrt{h(h+1)}} \frac{1}{\sqrt{2\pi s}} \{\cos(s - \tfrac{1}{4}\pi) - \sqrt{\pi\alpha}\cos(s+\alpha) +$$
$$+ 2\sqrt{\alpha}\,[S(\sqrt{\alpha})\cos(s+\alpha-\tfrac{1}{4}\pi) - C(\sqrt{\alpha})\sin(s+\alpha-\tfrac{1}{4}\pi)]\} \tag{3.19}$$

Let us deal with the case of weak shock waves ($\delta \to 1$). Here $\tanh\theta_0 = \beta \to 1$, so that $\theta_0 \to \infty$. From coefficients A_n in (2.15) there only remains $A_0 = -1/2\,v_0$. Then (2.21) yields

$$w(r, \theta) = v_0 J_1(r) \cosh\theta$$

From this we obtain the following formula for the pressure disturbance p', in terms of its undisturbed value p:

$$\frac{p'}{p} = -\frac{2(h+1)}{h}(M_0 - 1)k\Delta\exp(ikY)\frac{J_1[kct\sqrt{1-\tau^2}]}{\sqrt{1-\tau^2}} \qquad \left(\tau = \frac{X}{ct}\right) \tag{3.20}$$

The behavior of $\xi(s)$ when $\delta \to 1$ is easily determined if we bear in mind that $F(q+\theta_0) \to 0$ when $\theta_0 \to \infty$. As $a \to 1$, $b \to 0$, we find from (2.14)

$$w_2(q, \theta_0) \approx F(q - \theta_0) = -2v_0 \sinh\theta_0 e^{-2q}\cosh q$$

therefore

$$w_1(p, \theta_0) = -2v_0 \sinh\theta_0 (\sqrt{p^2+1} - p)^2$$

and using the tables of the originals and the reflections, we find

$$w(r, \theta_0) = -4v_0 \sinh\theta_0 \frac{J_2(r)}{r}$$

If we insert this into (2.5) for $\theta_0 \gg 1$ we find

$$\frac{\partial v}{\partial r} = -2v_0 \frac{J_2(r)}{r} \qquad \text{or} \qquad v(r, \theta_0) = v_0\left[1 - 2\int_0^s \frac{J_2(x)}{x}\, dx\right]$$

Making use of the well-known formulas

$$\frac{J_2(r)}{r} = -\frac{d}{dr}\left[\frac{J_1(r)}{r}\right], \qquad \int_0^\infty \frac{J_2(x)}{x}\, dx = \frac{1}{2}$$

we finally arrive at

$$\frac{v(r, \theta_0)}{v_0} = \frac{\xi(s)}{\Delta} = 2\frac{J_1(s)}{s} \quad \text{when } \delta \to 1$$

and this coincides with the corresponding formula in [1].

BIBLIOGRAPHY

1. Freeman, N.C., Theory of stability of the plane shock waves. *Proc. Roy. Soc.* A228, p. 341, 1955.

2. D'iakov, S.P., Ob ustoichivosti udarnykh voln (On the stability of shock waves). *Zh E T P* Vol. 27, No. 3 (9), p. 288, 1954.

3. Lavrent'ev, M.A. and Shabat, B.V., *Metody teorii funktsii kompleksnogo peremennogo (The Methods of the Theory of Functions of a Complex Variable)*. GTTI, 1951.

4. Gel'fond, A.O., *Ischeslenie konicheskykh raznostei (Calculation of Conical Differences)*. GTTI, 1952.

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